CS395T: Continuous Algorithms
Homework IV

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Due date: April 7, 2026, start of class (3:30 PM).

Please list all collaborators on the first page of your solutions. Unless we have discussed and I have specified otherwise, homework is not accepted if it is not turned in by hand at the start of class, or turned in electronically on Canvas by then. Send me an email to discuss any exceptions.

1 Problem 1

Let $\{\mathcal{T}_{\mathbf{x}}\}_{\mathbf{x} \in \mathbb{R}^d}$ be transition distributions corresponding to a random walk on $\mathbb{R}^d$ with stationary distribution $\pi^\star$, and let π be a density on $\mathbb{R}^d$ which is β-warm with respect to $\pi^\star$.

(i) Prove that $D_{\text{TV}}(\mathcal{T}\pi, \pi^\star) \le D_{\text{TV}}(\pi, \pi^\star)$.

(ii) Prove that $\mathcal{T}\pi$ is β-warm with respect to $\pi^\star$.

2 Problem 2

(i) Let $f : \Omega \to \mathbb{R}_{\ge 0}$ and $g : \Omega \to \mathbb{R}_{>0}$ for a sample space Ω satisfying $\frac{f(\omega)}{g(\omega)} \le C$ for all $\omega \in \Omega$, and let $P \propto f$ and $Q \propto g$ be probability distributions over Ω. Suppose we have sample access to Q through an oracle $\mathcal{O}$. Give an algorithm which samples from P, using at most

$$N := \frac{C \int_{\omega \in \Omega} g(\omega) \mathrm{d}\omega}{\int_{\omega \in \Omega} f(\omega) \mathrm{d}\omega}$$

queries to $\mathcal{O}$, as well as at most N value oracle queries to each of f and g, all in expectation.

(ii) Let $f : \mathbb{B}(\mathbf{0}_d, 1) \to \mathbb{R}$ be 1-Lipschitz. Give an algorithm which samples from the density $\pi \propto \exp(-f)$ over $\mathbb{B}(\mathbf{0}_d, 1)$, i.e.,

$$\pi(x) = \begin{cases} \frac{\exp(-f(\mathbf{x}))}{\int_{\mathbf{y} \in \mathbb{B}(\mathbf{0}_d, 1)} \exp(-f(\mathbf{y})) \mathrm{d}\mathbf{y}} & \mathbf{x} \in \mathbb{B}(\mathbf{0}_d, 1) \\ 0 & \text{else} \end{cases},$$

using a constant number of value oracle queries to f in expectation.[1]

3 Problem 3

Let $G = (V, E, \mathbf{w})$ be an undirected graph with Laplacian matrix $\mathbf{L}_G$ (Definition 5, Part II).

(i) What is the kernel of $\mathbf{L}_G$?

(ii) Give a combinatorial interpretation of the value of $\mathbf{x}^\top \mathbf{L}_G \mathbf{x}$, when $\mathbf{x} \in \{-\frac{1}{2}, \frac{1}{2}\}^V$.

(iii) Prove that there exists a graph $H = (V, E', \mathbf{w}')$ on the same vertex set V, with $E' \subseteq E$, $\mathbf{w}' \in \mathbb{R}^{E'}_{>0}$, and Laplacian matrix $\mathbf{L}_H$, such that $|E'| = O(|V| \log |V|)$, and[2]

$$0.9\mathbf{L}_G \preceq \mathbf{L}_H \preceq 1.1\mathbf{L}_G.$$

[1] This shows that exact sampling of log-Lipschitz functions over the unit ball can be done in an expected constant number of value oracle queries. In contrast, Lemma 4, Part II shows that even approximate optimization of Lipschitz functions over the unit ball requires exponentially many value oracle queries. This gives an example where a sampling problem is easier than the corresponding optimization problem, once we leave the convex setting.

[2] Section 3, Part IX may be helpful for this.

4 Problem 4

Let $f : \mathbb{R}^d \to \mathbb{R}$ be L-smooth and μ-strongly convex, and suppose we know $\mathbf{x}^\star := \text{argmin}_{\mathbf{x}\in\mathbb{R}^d} f(x)$. Let $\pi^\star : \mathbb{R}^d \to \mathbb{R}_{>0}$ be the density satisfying $\pi^\star(\mathbf{x}) \propto \exp(-f(\mathbf{x}))$. Give a density π_0 depending only on $\mathbf{x}^\star$, μ, and L, that is β-warm with respect to $\pi^\star$, for

$$\beta = \exp(O(d \log \kappa)), \text{ where } \kappa := \frac{L}{\mu}.$$

5 Problem 5

Let $\beta \geq 1$ and $\epsilon \in (0, 1)$, and let $\{\mathcal{T}_x\}_{x\in\mathbb{R}^d}$ be transition distributions corresponding to a random walk on $\mathbb{R}^d$ with stationary distribution $\pi^\star$. Assume that for any density π on $\mathbb{R}^d$ which is $\frac{\beta}{\epsilon}$-warm with respect to $\pi^\star$, $D_{\text{TV}}(\mathcal{T}\pi, \pi^\star) \leq \frac{1}{4}$. Prove that if π_0 is β-warm with respect to $\pi^\star$,[3]

$$D_{\text{TV}}\left(\mathcal{T}^k\pi_0, \pi^\star\right) \leq \epsilon \text{ if } k \geq \log_2\left(\frac{1}{\epsilon}\right).$$

[3] This gives a continuous generalization of Corollary 1, Part XIV.